

Theoretical model for negative giant magnetoresistance in ultra-high-mobility 2D electron systems.

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Abstract – We report on theoretical studies of the recently discovered negative giant magnetoresistance in ultraclean two-dimensional electron systems at low temperatures. We adapt a transport model to a ultraclean scenario and calculate the elastic scattering rate (electron-charged impurity) in a regime where the Landau level width is much smaller than the cyclotron energy. We obtain that for low magnetic fields the scattering rate and, as a consequence, the longitudinal magnetoresistance dramatically drop because of the small density of states between Landau levels. We also study the dependence of this striking effect on temperature and an in-plane magnetic field.

Introduction. – Electron transport assisted by external AC or DC fields in low-dimensional electron systems has been always a central topic in basic and applied research in Condensed Matter Physics. An important outcome is that in the last decade the quality and hence the mobility of two-dimensional electron systems (2DES), have been continuously increasing, exceeding routinely now the $10^7 \text{ cm}^2/\text{Vs}$ level. As a result of that, unexpected physical phenomena have been discovered such as, for example, the microwave-induced magnetoresistance (R_{xx}) oscillations (MIRO) and zero resistance states (ZRS). These effects were discovered when a 2DES in a low and perpendicular magnetic field (B) was irradiated with microwaves (MW) [1, 2]. Different theories have been proposed to explain these effects [3–8] but the physical origin still remains unclear. In the same way, a great effort has been also made from the experimental side [9–17].

An interesting and challenging experimental result, recently obtained [18, 19] and as intriguing as ZRS, consists in a strong resistance spike which shows up far off-resonance. It occurs at twice the cyclotron frequency, $w \approx 2w_c$ [18, 19], where w is the radiation frequency and w_c the cyclotron frequency. The amplitude of such a spike is very large reaching an order of magnitude regarding MIRO. Another remarkable result in the same experiments is a dramatic drop in the magnetoresistance, in other words, it is obtained negative giant magnetoresistance (NGMR) confined at low B , ($B \leq 0.1T$). The ap-

pearance of this effect is concurrent with the off-resonance magnetoresistance spike and always in very high mobility samples. In all previous experiments about MIRO and ZRS, using lower mobility samples, this concurrence was never obtained. Therefore, we must conclude that, in some way, the two physical phenomena have to be connected or share the same physical origin. The first experiment to obtain NGMR, (without irradiation) was carried out by [Paalanen et al \[20\]](#) and later on by [Bockhorn et al. \[21\]](#), where they study the dependence of NGMR on temperature and electron density. Next, [Y. Dai et al. \[22\]](#), reported on the dependence of NGMR on an in-plane B . More recently [Hatke et al. \[23\]](#), obtained experimental results on the dependence of NGMR on temperature and a tilted B . Finally, the most recent experimental results on NGMR are by [Mani et al.](#), where they report on the dependence of NGMR on the sample size [24]. One important outcome, common to all experiments, is that when increasing the sample disorder, NGMR tends to progressively disappear. In this way, higher temperatures, more intense in-plane B and bigger sample size, all of them contribute to increase the disorder, giving rise to a vanishing NGMR. On the theoretical side, although some works have been published on the magnetoresistance spike [25, 26], however no theoretical approach trying to explain the physical origin of NGMR or its connection with the off-resonance magnetoresistance spike, has been presented to date.

In this article, we theoretically study and discuss the physical origin of NGMR and its dependence on temper-

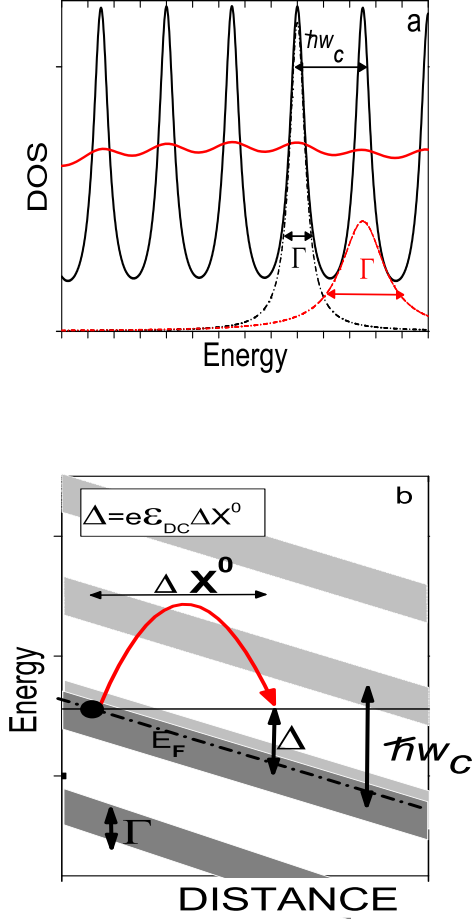


Fig. 1: a) Schematic diagram showing the density of Landau states simulated by Lorentzian functions for wide and narrow Landau levels; the width of the states is indicated by Γ . b) Schematic diagram presenting the elastic scattering process for ultraclean samples. The narrow Landau levels are presented by stripes of grey color. They are tilted by the action of the static electric field ξ_{DC} in the current direction (x -direction). The subsequent energy drop in the scattering jump is given by $\Delta = e\xi_{DC}\Delta X^0$.

ature and an in-plane B . We extend a previous transport model for 2DES based in elastic scattering between Landau levels (LL) due to charged impurities [3, 27]. This transport model was developed by the authors to deal with MIRO and ZRS [3, 4, 28, 29]. We adapt this model to ultraclean samples, obtaining that the scattering conditions are strongly modified. Mainly because the LL, which in principle are broadened by scattering, become very narrow in this kind of samples. This implies an increasing number of states at the center of the LL sharing a similar energy. However, in between LL, it happens the opposite, the density of states dramatically decreases (see Fig.1). We show that, at low B and for the standard DC static electric field used in these experiments, the final density of states in the

elastic scattering process, corresponds to a region between LL, therefore with a very low density of states. This gives rise to a small scattering rate and eventually an important drop in the measured current and R_{xx} . The result of an increasing temperature or an in-plane magnetic field is to make bigger the disorder of the sample increasing, in turn, the LL width. As a result there will be more available states between LL, resulting in a stronger scattering rate and a bigger R_{xx} . The final outcome is that NGMR tends to vanish.

Theoretical model. – In our model of transport we basically follow the approach by Ridley [27] and calculate first the scattering suffered by the electrons due to charged impurities (elastic) applying time dependent first order perturbation theory. Thus, we calculate the scattering rate [3, 28] between two Landau states (LS), the initial, n , and the final, m with the Fermi's Golden Rule:

$$W_I = \frac{2\pi}{\hbar} |\langle \phi_m | V_s | \phi_n \rangle|^2 \delta(E_n - E_m) \quad (1)$$

where ϕ_n and ϕ_m are the wave functions corresponding to the initial and final LS respectively, i.e., they represent quantum harmonic oscillators. V_s is the scattering potential for charged impurities [30], $V_s = \sum_q \frac{e^2}{2S\epsilon(q+q_s)} \cdot e^{i\vec{q} \cdot \vec{r}}$ S being the surface of the sample, ϵ the GaAs dielectric constant, and q_s is the Thomas-Fermi screening constant [30, 31]. $E_n = \hbar w_c(n + 1/2)$ and $E_m = \hbar w_c(m + 1/2) - \Delta$ are the corresponding LS energies for the initial and final states respectively. Δ is the energy drop along the scattering jump up to the final LS due to static electric field ξ_{DC} . ξ_{DC} is aligned with the x direction and is, in turn, responsible of the current (see Fig. 1). A more elaborated expression for the scattering rate can be obtained [3, 27] being given by:

$$W_I = \frac{e^4 n_i}{8\pi^2 \hbar \epsilon^2} \sum_m \left[\frac{\Gamma}{[E_n - E_m]^2 + \Gamma^2} \right] \times \int d\theta \int dq \frac{q}{(q + q_s)^2} e^{-\frac{1}{2} q^2 R^2} \left[L_n \left(\frac{1}{2} q^2 R^2 \right) \right]^2$$

where n_i is the impurity density, R is the magnetic characteristic length, $R^2 = \frac{\hbar}{eB}$. L_n are the associated Laguerre polynomials. In the obtained expression for the impurity scattering rate, the delta function, $\delta(E_n - E_m)$, has been approached by a Lorentzian, considering that the LL are broadened by disorder,

$$\delta(E_n - E_m) \simeq \frac{1}{\pi} \frac{\Gamma}{(E_n - E_m)^2 + \Gamma^2} \quad (3)$$

where Γ is the LL width. On the other hand, the sum is carried out up to all final LS, ($\sum_{m=0}^{\infty}$).

When it comes to extending the transport model to a ultraclean scenario it is essential to consider that now the LL width is much smaller than $\hbar w_c$. Accordingly, we first apply the Poisson sum rules to perform the infinite sum

of LL in eq. (2) and obtain:

$$\sum_m \left[\frac{\Gamma}{[E_n - E_m]^2 + \Gamma^2} \right] = \frac{1}{\hbar w_c} \left\{ 1 + 2 \sum_{s=1}^{\infty} \cos \left[\frac{2\pi s \Delta}{\hbar w_c} \right] e^{\left[-\frac{\pi \Gamma s}{\hbar w_c} \right]} \right\} \quad (4)$$

When $\Gamma \ll \hbar w_c$, (ultra-clean scenario), is highly recommended, if possible, to carry out the total sum over s inside the curly brackets [33]:

$$\begin{aligned} \sum_{s=1}^{\infty} \left[\cos \left[\frac{2\pi s \Delta}{\hbar w_c} \right] \exp \left[-\frac{\pi \Gamma s}{\hbar w_c} \right] \right] &= \\ &= \frac{e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \left\{ \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] - e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \right\}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \end{aligned} \quad (5)$$

to finally obtain,

$$\begin{aligned} \sum_m \left[\frac{\Gamma}{[E_n - E_m]^2 + \Gamma^2} \right] &= \\ \frac{1}{\hbar w_c} \left\{ \frac{1 - e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \right\} \end{aligned} \quad (6)$$

Then, substituting this result into W_I we get to:

$$W_I \propto \left\{ \frac{1 - e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \right\} \quad (7)$$

Once we know the scattering rate, we consider that when an electron undergoes a scattering process, due to charged impurities, its average orbit center position changes in the static electric field direction, (ξ_{DC} or x direction), from X_0 to X'_0 . Accordingly, it advances an average effective distance given by [3, 25]: $\Delta X^0 = X'_0 - X_0 \simeq 2R_c$ [32], R_c being the orbit radius $R_c = \sqrt{2m^* E_F} / eB$. Therefore, Δ and ξ_{DC} are related by $\Delta = e\xi_{DC} \Delta X^0$.

Now, we can obtain the expression for the longitudinal conductivity σ_{xx} according to [3, 4]: $\sigma_{xx} \propto \int dE \frac{\Delta X^0}{\tau}$ being E the energy and $\tau = \frac{1}{W_I}$ the scattering time. To obtain R_{xx} we use the relation $R_{xx} = \frac{\sigma_{xx}}{\sigma_{xx} + \sigma_{xy}} \simeq \frac{\sigma_{xx}}{\sigma_{xy}}$, where $\sigma_{xy} \simeq \frac{n_e e}{B}$, being n_e the electron density, and $\sigma_{xx} \ll \sigma_{xy}$. Finally, the expression of R_{xx} reads:

$$R_{xx} \propto \left\{ \frac{1 - e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \right\} \quad (8)$$

Thus, R_{xx} directly depends on Γ and ξ_{DC} . For the experimental parameters that we are dealing with [21–23], we have estimated that $\xi_{DC} \sim 1 - 2$ V/m. On the other hand for low $B \sim 0.04 - 0.05$ T, we have obtained that, in average, in the advanced distance corresponding

to a scattering jump, the LL are tilted an energy $\Delta \simeq 3.10^{-5}$ eV. For these small B the cyclotron energy, $\hbar w_c \sim 7 - 8 \times 10^{-5}$ eV. Then, comparing both numerical values ($\frac{\Delta}{\hbar w_c} \sim \frac{1}{2}$), we can conclude that in ultraclean samples and low B , the average scenario is the corresponding to an electron "landing", after a scattering event, between LL where there is a low density of states (see Fig. 1.b). The result is a dramatic drop at low B in R_{xx} as obtained in the experiments. Thus, when it is fulfilled that, $\frac{\Delta}{\hbar w_c} \sim \frac{1}{2}$, the first term between brackets tends to:

$$\left\{ \frac{1 - e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \right\} \rightarrow \left\{ \frac{1 - e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]}}{1 + e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]}} \right\} \quad (9)$$

and if, in addition to that, $\Gamma \ll \hbar w_c$, for low B the resulting term decreases very much affecting R_{xx} which becomes also very small, producing the effect of NGMR. However, when increasing further B , it turns out that, $\frac{\Delta}{\hbar w_c} \rightarrow 0$, and then:

$$\left\{ \frac{1 - e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}}{1 - 2e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] + e^{\left[-\frac{2\pi \Gamma}{\hbar w_c} \right]}} \right\} \rightarrow \left\{ \frac{1 + e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]}}{1 - e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]}} \right\} \quad (10)$$

and R_{xx} tends to increase with increasing B .

In a non-ultraclean sample where $\Gamma \geq \hbar w_c$, the sum over final LS can be written as:

$$\sum_m \left[\frac{\Gamma}{[E_n - E_m]^2 + \Gamma^2} \right] \simeq \frac{1}{\hbar w_c} \left\{ 1 + 2 \cos \left[\frac{2\pi \Delta}{\hbar w_c} \right] e^{\left[-\frac{\pi \Gamma}{\hbar w_c} \right]} \right\} \quad (11)$$

where the cosine term is strongly damped by the exponential, preventing the appearance of NGMR.

Results. – As we said above, the effect of both, temperature and an in-plane B is to increase the disorder of the sample with the subsequent increase of Γ . The bigger the disorder, the wider Γ , increasing the density of states between LL. The outcome is a stronger elastic scattering rate and eventually an increasing R_{xx} and vanishing NGMR. In the case of an increasing temperature, electrons are able to interact more strongly with the ions in the lattice, giving rise to a stronger emission of acoustic phonons and scattering rate. This has to be reflected in the total quantum scattering rate $1/\tau_0$, that encompasses all scattering sources. According to the Matthiessen rule the total scattering rate can be expressed as the sum of the different individual scattering sources, $\frac{1}{\tau_0} = \sum_i \frac{1}{\tau_i}$ and obviously one of them is the acoustic phonon scattering rate. Then, an increase in the phonon scattering rate ($\gamma_{ac} = 1/\tau_{ac}$) due to temperature will eventually make the total scattering rate to increase too. According to Ando et al [30] γ_{ac} depends linearly on T : $\gamma_{ac} \propto T$ and then $\Delta\gamma_{ac} \propto \Delta T$. This will be reflected in the final LL width, Γ_f , that can be expressed as: $\Gamma_f = \Gamma_i + \hbar\Delta\gamma_{ac}$, where Γ_i is the initial LL width corresponding to the initial temperature. The

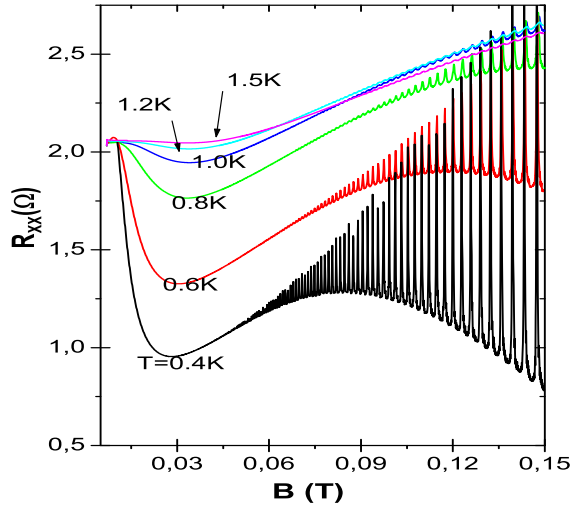


Fig. 2: Calculated longitudinal magnetoresistance, R_{xx} , vs magnetic field for several temperatures ranging from 0.4 K to 1.5 K. It is clearly observed the vanishing effect of the increasing temperature on the NGMR.

effect of temperature is presented in Fig. 2, which exhibits R_{xx} as a function of B for several temperatures ranging from 0.4 K to 1.5 K. It is clearly observed that an increasing temperature makes NGMR to progressively disappear. When reaching $T > 1$ K, NGMR is totally wiped out.

The effect of an in-plane magnetic field, (B_x), on the transport in a 2DES was studied in experiments on radiation-induced resistance oscillations [34, 35]. Experimental results showed that the main effect was a progressive damping of the whole resistance response as B_x increased. Subsequent theoretical results [36], confirmed and explained the surprising damping in the framework of the radiation-driven electrons orbits model: the presence of B_x imposes an extra harmonically oscillating motion in the z -direction enlarging the electrons trajectory in their cyclotron orbits (see Fig. 3). This would increase the interactions of electrons with the lattice and with the walls of the quantum well giving rise of a stronger emission of acoustic phonons. Therefore, the effect of the presence of B_x is to increase the disorder in the sample and the width of the LL. The relation between γ_{ac} and B_x is given by [36]:

$$\gamma_{ac} = \gamma_{ac}(B_x = 0) \times \sqrt{1 + \left(\frac{eB_x z_0^2}{\hbar}\right)^2} \quad (12)$$

where z_0 is the effective length of the electron wave function when we consider a parabolic potential for the z -confinement [30, 31]. Now, proceeding similarly as before with temperature, we can express the final width of LL as, $\Gamma_f = \Gamma_i(B_x = 0) + \hbar\Delta\gamma_{ac}$, where in this case $\Delta\gamma_{ac} = \gamma_{ac}(B_x \neq 0) - \gamma_{ac}(B_x = 0)$. The effect of B_x

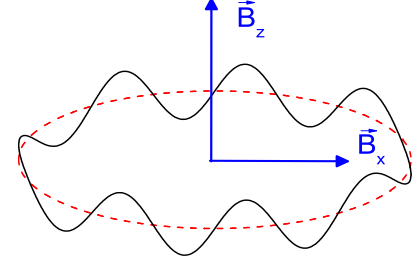


Fig. 3: Schematic diagram showing the semiclassical description of electron trajectories in 2D systems in the presence of a perpendicular B_z , and a parallel B_x .

is presented in Fig. 4, where we exhibit R_{xx} as a function of B for several B_x ranging from 0. T to 1.0 T. We observe that for $B_x \simeq 1$ T, NGMR totally disappears. It has been also recently reported by Mani et al. [24] on the effect of sample size on NGMR. They report that NGMR is more pronounced in smaller samples and that the effect progressively disappears as the sample size increases. The explanation can be readily obtained, at least qualitatively, with similar terms as temperature and B_x . In this case smaller samples present in average a weaker scattering and the electron transport gets closer to quasi-ballistic. Therefore, in this kind of samples Γ will be much smaller presenting a clear NGMR. Increasing the sample size it is expected that R_{xx} will increase and NGMR will disappear, as experiments report.

Conclusions. — In summary, we have reported, from a theoretical approach, on the recently discovered NGMR in ultraclean 2DES. We adapt a transport model to high mobility samples and calculate the elastic scattering rate in a regime where the Landau level width is much smaller than the cyclotron energy. We obtain that for low B the scattering rate and R_{xx} dramatically drop because there are very few available states where to get to. We also study the dependence of this striking effect on temperature and an in-plane magnetic field, and conclude that both of them increase the disorder of the sample giving rise to a bigger Γ and stronger scattering rate. The subsequent results is a greater R_{xx} and a vanishing NGMR.

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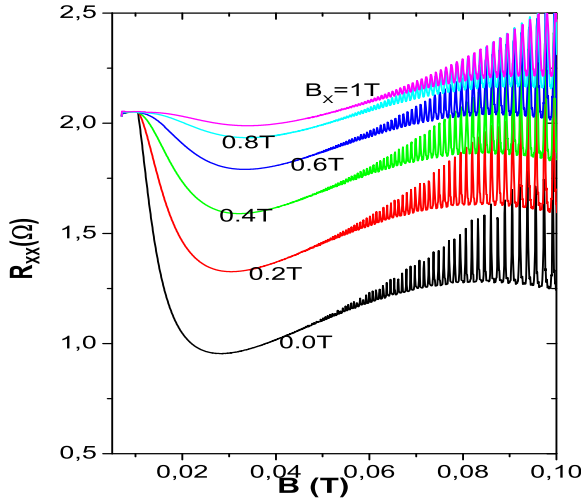


Fig. 4: Calculated longitudinal magnetoresistance, R_{xx} , vs magnetic field for different values of the in-plane magnetic field, (B_x). B_x is ranging from 0 T to 1 T.

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